

## APPENDIX 19.A: FORMULAS FOR GEOMETRIC AVERAGE OPTIONS

The discussion of Monte Carlo valuation of geometrically averaged Asian options enables us to understand the formulas for geometrically averaged Asian options, described in Appendix 14.A. If we sample the stock price  $N$  times from time 0 to  $T$ , with the distance between samples  $h = T/N$  and the first sample occurring at time  $h$ , the log of the geometric average is

$$\begin{aligned} \frac{1}{N} \sum_{i=1}^N \ln(S_{ih}) &= \frac{1}{N} \left[ \sum_{i=1}^N \left( \ln(S_0) + (r - \delta - 0.5\sigma^2)ih + \sigma \sum_{j=1}^i Z_j \sqrt{h} \right) \right] \\ &= \ln(S_0) + (r - \delta - 0.5\sigma^2) \frac{h}{N} \sum_{i=1}^N i + \frac{\sigma \sqrt{h}}{N} \sum_{i=1}^N \sum_{j=1}^i Z_j \end{aligned} \quad (19.18)$$

where  $Z_j \sim \mathcal{N}(0, 1)$  and  $Z_i$  and  $Z_j$  are independent.

The last double summation can be rewritten as

$$\sum_{i=1}^N \sum_{j=1}^i Z_j = NZ_1 + (N-1)Z_2 + \cdots + Z_N$$

Thus we have

$$\begin{aligned} E \left[ \frac{1}{N} \sum_{i=1}^N \ln(S_{ih}) \right] &= \ln(S_0) + (r - \delta - 0.5\sigma^2) \frac{h}{N} \frac{N(N+1)}{2} \\ &= \ln(S_0) + (r - \delta - 0.5\sigma^2) T \frac{N+1}{2N} \end{aligned}$$

where we have used the fact that

$$\sum_{i=1}^N i = \frac{N(N+1)}{2}$$

The variance is

$$\begin{aligned} \text{Var} \left[ \frac{1}{N} \sum_{i=1}^N \ln(S_{ih}) \right] &= \frac{\sigma^2 h}{N^2} (N^2 + (N-1)^2 + \cdots + 1) \\ &= \sigma^2 T \frac{N(N+1)(2N+1)}{6N^3} \end{aligned}$$

where we have used the fact that

$$\sum_{i=1}^N i^2 = \frac{N(N+1)(2N+1)}{6}$$

These calculations tell us that the average price,  $G(T)$ , can be written as a lognormal process,

$$G(T) = S_0 e^{\left[ (r-\delta-0.5\sigma^2) \frac{T}{2} \frac{N+1}{N} + \sigma \sqrt{T} \sqrt{\frac{(N+1)(2N+1)}{6N^2}} Z \right]}$$

where  $Z \sim \mathcal{N}(0, 1)$ . Using equation (18.13), we have

$$E[G(T)] = S_0 e^{\left[ (r-\delta-0.5\sigma^2) \frac{N+1}{N} + \sigma^2 \frac{(N+1)(2N+1)}{6N^2} \right] \frac{1}{2} T}$$

The prepaid forward price for the average is

$$e^{-rT} E[G(T)] = S_0 e^{-\left[ r \frac{N-1}{N} + (\delta+0.5\sigma^2) \frac{N+1}{N} - \sigma^2 \frac{(N+1)(2N+1)}{6N^2} \right] \frac{1}{2} T}$$

Thus, we can price an option on the geometric average by setting the dividend yield for the average,  $\delta^*$ , equal to

$$\delta^* = \frac{1}{2} \left[ r \frac{N-1}{N} + (\delta+0.5\sigma^2) \frac{N+1}{N} - \sigma^2 \frac{(N+1)(2N+1)}{6N^2} \right]$$

and the volatility of the average,  $\sigma^*$ , equal to

$$\sigma^* = \frac{\sigma}{N} \sqrt{\frac{(N+1)(2N+1)}{6}}$$

If we take the limit as  $N \rightarrow \infty$ , we have

$$\delta^* = \frac{1}{2} \left( r + \delta + \frac{1}{6} \sigma^2 \right)$$

$$\sigma^* = \sigma \sqrt{\frac{1}{3}}$$

## Average Price Options

Geometric average price options, for which the geometric average replaces the stock price, are priced by substituting  $\delta^*$  and  $\sigma^*$  for  $\delta$  and  $\sigma$  in the Black-Scholes formula.

## Average Strike Options

With geometric average strike options, the average replaces the strike price. Thus, we replace the risk-free rate with  $\delta^*$  and the strike price with  $S_0$ . However, we also need to compute the volatility of the *difference* between  $\ln(S_T)$  and  $\ln(A(T))$ ,

$$\sigma^{**2} = \text{Var}[\ln(S_T) - \ln(A_T)]$$

We can write

$$\ln(S_T) = \ln(S_0) + (r - \delta - 0.5\sigma^2)T + \sigma \sum_{i=1}^N Z_i \sqrt{h}$$

Using equation (19.18), the covariance between  $\ln(S_T)$  and  $\ln(A_T)$  is

$$\begin{aligned} E \left[ \left( \sigma \sqrt{h} \sum_{i=1}^N Z_i \right) \left( \frac{\sigma \sqrt{h}}{N} \sum_{i=1}^N \sum_{j=1}^i Z_j \right) \right] &= \frac{\sigma^2 h}{N} \left( \sum_{i=1}^N Z_i \right) \left( \sum_{i=1}^N \sum_{j=1}^i Z_j \right) \\ &= \frac{\sigma^2 h}{N} (N + (N-1) + \dots + 1) \\ &= \frac{\sigma^2 h}{2} (N+1) \end{aligned}$$

The correlation coefficient,  $\rho$ , is therefore

$$\begin{aligned} \rho &= \frac{\frac{\sigma^2 h}{2} (N+1)}{\left( \sigma \sqrt{T} \right) \left( \sigma \sqrt{T} \frac{1}{N} \sqrt{\frac{(N+1)(2N+1)}{6}} \right)} \\ &= \frac{1}{2} \sqrt{\frac{6(N+1)}{2N+1}} \end{aligned}$$

Note that when  $N = 1$ ,  $\rho = 1$ , and as  $N \rightarrow \infty$ ,  $\rho = \sqrt{3}/2$ .

Using this expression for  $\rho$ , we have

$$\sigma^{**2} = \sigma^2 T + \sigma^2 T \frac{(N+1)(2N+1)}{6N^2} - 2\rho\sigma^2 T \frac{1}{N} \sqrt{\frac{(N+1)(2N+1)}{6}}$$

Thus, to value an average strike option we substitute  $\sigma^{**}$  for the volatility and  $\delta^*$  for the interest rate. The dividend yield on the underlying asset remains the same.