APPENDIX 21.C: SOLUTIONS FOR PRICES AND PROBABILITIES

The Black-Scholes partial differential equation has the form:

$$V_t + \tfrac{1}{2}\sigma^2 S^2 V_{SS} + \eta S V_S = \beta V \tag{21.46}$$

In equation (21.11), we have $\eta = r - \delta$ and $\beta = r$. When $\beta = 0$, equation (21.46) is the backward equation, equation (21.31).

Suppose we guess the following general solution to equation (21.46):

$$V(S,t) = Ae^{\gamma t} S^a N(x)^y \tag{21.47}$$

$$x = \frac{ln[S(t)] + f + g(T-t)}{\sigma\sqrt{T-t}}$$

where $A, a, f, g,$ and γ are constants to be determined, and σ and y are parameters. $N(x)$ is the cumulative standard normal distribution. We will consider the cases $y = \{0, 1\}$. Note that sums of solutions are also solutions.

Computing the various derivatives of this guessed solution, substituting them into equation (21.46), and simplifying, gives

$$\begin{aligned} 0 = {} & \left[\tfrac{1}{2}\sigma^2 a^2 + a\left(\eta - \tfrac{1}{2}\sigma^2\right) + \gamma - \beta\right] \\ & + yN(x)^{-1}N'(x)\left[\frac{\sigma^2\left(a - \frac{1}{2}\right) + \eta - g}{\sigma\sqrt{T-t}}\right] \end{aligned} \tag{21.48}$$

The parameters A and f are not in any way determined by this equation; hence, they are solely determined by boundary conditions. Equation (21.48) is satisfied for

$$a = \left(\tfrac{1}{2} - \frac{\eta}{\sigma^2}\right) \pm \sqrt{\left(\frac{\eta}{\sigma^2} - \tfrac{1}{2}\right)^2 + 2\frac{\beta - \gamma}{\sigma^2}} \tag{21.49}$$

and

$$g = \sigma^2 \left(a - \tfrac{1}{2}\right) + \eta \tag{21.50}$$

The first term in square brackets stems from differentiating $Ae^{\gamma t} S^a$, while the second bracketed term stems from differentiating $N(x)$. We will examine only a few of the commonly occurring solutions. Since $Ae^{-\gamma(T-t)} S^a N(x)$ and $Ae^{-\gamma(T-t)} S^a$ both solve the PDE, and since sums of solutions are also solutions, then

$$Ae^{-\gamma(T-t)} S^a \left[N(x) - 1\right] = Ae^{-\gamma(T-t)} S^a N(-x)$$

is also a solution.

Solutions to the Black-Scholes Equation

The parameters η and β are determined by the PDE that arises in solving a particular problem. In the standard Black-Scholes equation, $\eta = r - \delta$ and $\beta = r$; this is the case we will consider. Let a^+ denote the positive root in equation (21.49), and a^- the negative root. Since g is defined in terms of a, for any given γ, there are two matched $\{a, g\}$ pairs.

If we pick γ, the rest of the solution is determined by equations (21.49) and (21.50) in conjunction with boundary conditions. Two obvious choices are $\gamma = r$ and $\gamma = \delta$. If $\gamma = r$, then $\{a^+, g^+\} = \{0, r - \delta - \frac{1}{2}\sigma^2\}$ and $\{a^-, g^-\} = \left\{1 - 2\frac{r-\delta}{\sigma^2}, -\left(r - \delta - \frac{1}{2}\sigma^2\right)\right\}$. The positive roots here, together with appropriate boundary conditions, generate the price of a cash-or-nothing option, equation (21.16). If $\gamma = \beta - \eta = \delta$, then $\{a^+, g^+\} = \{1, r - \delta + \frac{1}{2}\sigma^2\}$ and $\{a^-, g^-\} = \left\{-2\frac{r-\delta}{\sigma^2}, -\left(r - \delta + \frac{1}{2}\sigma^2\right)\right\}$. The positive roots here, together with boundary conditions, generate the price of an asset-or-nothing option, equation (21.15).

The following expressions all satisfy the Black-Scholes PDE:

$$V^5[S(t), t] = e^{-\delta(T-t)} S^{-a_3} \times N\left(\frac{ln[S(t)] + f - [r - \delta + 0.5\sigma^2][T - t]}{\sigma\sqrt{T - t}}\right) \tag{21.51}$$

$$V^6[S(t), t] = e^{-r(T-t)} S^{1-a_3} \times N\left(\frac{ln[S(t)] + f - [r - \delta - 0.5\sigma^2][T - t]}{\sigma\sqrt{T - t}}\right) \tag{21.52}$$

$$V^7[S(t), t] = AS(t)^{a_1} \tag{21.53}$$

$$V^8[S(t), t] = AS(t)^{a_2} \tag{21.54}$$

TABLE 21.2 Parameters generating the solutions to the Black-Scholes PDE for the indicated equation.

Equation	$\boldsymbol{y}$	$\boldsymbol{\gamma}$	$\boldsymbol{a}$ **(Equation (21.49))**
(21.12)	0	r	$a^+ = 0$
(21.13)	0	δ	$a^+ = 1$
(21.15)	1	δ	$a^+ = 1$
(21.16)	1	δ	$a^- = -2\frac{r-\delta}{\sigma^2}$
(21.51)	1	r	$a^+ = 0$
(21.52)	1	r	$a^- = 1 - 2\frac{r-\delta}{\sigma^2}$
(21.53)	0	0	a^+
(21.54)	0	0	a^-

where:

$$a_1 = \left(\tfrac{1}{2} - \frac{r-\delta}{\sigma^2}\right) + \sqrt{\left(\frac{r-\delta}{\sigma^2} - \tfrac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}$$

$$a_2 = \left(\tfrac{1}{2} - \frac{r-\delta}{\sigma^2}\right) - \sqrt{\left(\frac{r-\delta}{\sigma^2} - \tfrac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}$$

$$a_3 = \frac{2(r-\delta)}{\sigma^2}$$

With appropriate choice of A, equations (21.53) and (21.54) are the formulas for infinitely lived options. We will see in Chapter 22 that equations (21.51) and (21.52) play a role in pricing barrier options.

The solutions to the equations in Table 21.2 are obtained by choosing the parameters listed there. These equations also satisfy specific boundary conditions.

Solutions to the Backward Equation

For a stock following geometric Brownian motion, the backward equation is satisfied if $\beta = 0$. It turns out that $\gamma = 0$ is frequently the solution of interest. For example, the Black-Scholes term $N(d_2)$, without a discount factor, is the risk-neutral probability that $S(T) > K$, and is a solution to the forward equation.

Consider these (undiscounted) variants of equations (21.51) and (21.52):

$$e^{r(T-t)}V^5[S(t),t] = e^{(r-\delta)(T-t)}S(t)^{-a_3}N\left(\frac{ln[S(t)] + f - [r-\delta+0.5\sigma^2][T-t]}{\sigma\sqrt{T-t}}\right) \tag{21.55}$$

$$e^{r(T-t)}V^6[S(t), t] = S^{1-a_3}N\left(\frac{ln[S(t)] + f - [r - \delta - 0.5\sigma^2][T - t]}{\sigma\sqrt{T - t}}\right) \tag{21.56}$$

You can verify that equations (21.55) and (21.56) obey equation (21.32). With an appropriate scale factor and choice of f, equation (21.56) will appear in Chapter 22 as the risk-neutral probability that the stock price hits a barrier and exceeds a terminal strike price.

Finally, note that you can use Proposition 21.1 to obtain equation (21.55) from equation (21.56).