The method of solving constrained extremum problems devised by Lagrange is appropriate if the constraints hold with strict equality. This method works even when the constraint need not hold with equality in general, as long as we know that it will hold with equality at the solution to the problem. For example, even if Lisa, who would always like to consume more goods, doesn’t have to spend all of her money, we know that she will. However, if we do not know whether a constraint will be satisfied with equality, we need new tools.

The figure illustrates the distinction between an unconstrained maximum and a maximum for a concave objective function $f(x)$ subject to an inequality constraint. The unconstrained function reaches a maximum at its peak, point $a$, where a line tangent to the curve is horizontal. That is, the first-order condition requires that $\frac{df(x)}{dx} = 0$.

If $x$ is constrained to be less than or equal to $z$, $x \leq z$, then point $b$ in the figure is the constrained maximum. It occurs where the vertical constraint line at $z$ intersects the function. There the line tangent to the function, or first-order condition, is upward sloping, so $\frac{df(x)}{dx} > 0$. 

**CA1.1 Maximizing with Inequality Constraints**
An inequality constraint need not bind. If \( z \) is so large that it exceeds the \( x \) corresponding to point \( a, x_a \), then the inequality constraint does not bind and the maximum remains at \( a \), where the unconstrained-maximum first-order condition holds. We can solve these types of problems mathematically by using the Kuhn-Tucker method, named after its inventors, Harold Kuhn and Albert Tucker. We start by applying the method to a specific example and then use it on a general problem.

**An Illustration of the Kuhn-Tucker Method**

The Kuhn-Tucker approach closely resembles the Lagrange approach except that it permits the use of inequality (”greater-than-or-equal-to”) constraints as well as equality constraints. To illustrate this method, we consider the problem of trying to maximize an objective function \( a \ln(x_1) + b \ln(x_2) \), where \( a \) and \( b \) are positive, subject to the inequality constraints that \( z - p_1 x_1 - p_2 x_2 \geq 0 \) and \( x_1 \geq 0 \), where \( p_1, p_2, \) and \( z \) are all positive. It is possible that these constraints could hold with equality. For example, it is possible that the solution to this problem involves setting \( x_1 \) equal to zero. We write this problem as

\[
\max_{x_1, x_2} a \ln(x_1 + 1) + b \ln(x_2) \\
\text{s.t. } z - p_1 x_1 - p_2 x_2 \geq 0, \quad x_1 \geq 0.
\]

(A.19)

The collection of all the constraints on choice variables implicitly defines a set of “feasible” values for the choice variables. In the present example, the set of feasible values is defined by \( \{(x_1, x_2) | z - p_1 x_1 - p_2 x_2 \geq 0 \text{ and } x_1 \geq 0\} \), called the constraint set.

We now formulate the Lagrangian function (the function is still named after Lagrange rather than after Kuhn and Tucker) by choosing some additional variables to multiply times the left-hand side of the constraints, and then adding these to the objective function,

\[
\mathcal{L}(x_1, x_2; \lambda, \mu) = \ln(x_1 + 1) + \ln(x_2) + \lambda \left( y - p_1 x_1 - p_2 x_2 \right) + \mu x_1,
\]

where \( \lambda \) and \( \mu \) are called the Kuhn-Tucker multipliers (or often simply multipliers).

Kuhn and Tucker showed that we can characterize the solution to problem A.19 using four conditions (two sets of two conditions each). The first two equations are the first-order conditions that are obtained by setting the partial derivatives of the Lagrangian function with respect to the original choice variables, \( x_1 \) and \( x_2 \), equal to zero:

\[
\frac{\partial \mathcal{L}}{\partial x_1} = \frac{a}{x_1 + 1} - p_1 \lambda + \mu = 0, \tag{A.20}
\]

\[
\frac{\partial \mathcal{L}}{\partial x_2} = \frac{b}{x_2} - p_2 \lambda = 0. \tag{A.21}
\]

The next two conditions, called complementary slackness conditions, state that the product of each multiplier and the left-hand side of the corresponding constraint equals zero:

\[
\lambda \left( y - p_1 x_1 - p_2 x_2 \right) = 0, \tag{A.22}
\]
That is, either the constraint holds with equality or the multiplier is zero.

To find the solution to the problem A.19, we solve Equations A.20–A.23 in several steps. Combining the first-order conditions in Equations A.20 and A.21 with the complementary slackness conditions in Equations A.22 and A.23 gives us a system of equations that characterize any local extrema for the problem, provided that both objective function and constraints are all continuously differentiable in the choice variables.

Rearranging Equation A.21, we find that
\[ \lambda = \frac{b}{p_2 x_2}, \]
so because \( b \) and \( p_2 \) are positive, \( \lambda \) is strictly positive: \( \lambda > 0 \). Combining this result with the first-order condition for \( x_1 \), Equation A.22, we find that the first constraint holds with equality:
\[ z - p_1 x_1 - p_2 x_2 = 0. \]
Moreover, by substituting this expression for \( \lambda \) into the first-order condition for \( x_2 \), Equation A.21, we obtain
\[
\frac{a}{x_1 + 1} + \mu = b \frac{p_1}{p_2 x_2}.
\]
Multiplying both sides of this expression by \((x_1 + 1)\) yields
\[
a + \mu x_1 + \mu = b(x_1 + 1) \frac{p_1}{p_2 x_2}.
\]
However, \( \mu x_1 = 0 \) from Equation A.23, so we know that
\[
(a + \mu)p_2 x_2 = b p_1 (x_1 + 1). \tag{A.24}
\]

Now we have two cases to consider. Either \( x_1 \) or \( \mu \) must be zero if Equation A.23 is to be satisfied. If \( \mu = 0 \) and we substitute that value into Equation A.24, we find that
\[
x_2 = \frac{p_1 b}{p_2 a}(x_1 + 1).
\]
Substituting this expression into the complementary slackness condition for the first constraint, Equation A.22, and remembering that \( \lambda > 0 \), we find that
\[
x_2 = \frac{b}{a + b p_2}, \tag{A.25}
\]
\[
x_1 = \frac{a}{a + b p_1} z - 1.
\]

Now instead suppose that \( x_1 = 0 \), so Equation A.24 becomes
\[
x_2 = \frac{p_1 b}{p_2 a + \mu}.
\]
Remembering that \( \lambda > 0 \) and substituting this expression into the complementary slackness condition for the first constraint, Equation A.22, we find that \( \mu = p_1 (b/y) - a \) and \( x_2 = y/p_2 \).
Thus we have two possible solutions. Either

\[ x_1 = \frac{a}{a + b} \frac{z}{p_1} - 1, \quad x_2 = \frac{b}{a + b} \frac{z}{p_2}, \quad \text{and} \quad \mu = 0; \text{or} \]  
\[ x_1 = 0, \quad x_2 = \frac{z}{p_2}, \quad \text{and} \quad \mu = p_1 \frac{b}{z} - a. \]  

This multiplicity of possible solutions, Equations A.26 and A.27, does not mean that both solve the maximization problem. Only one of these possible answers solves the maximization problem, and which one is the solution depends on the values of the parameters \(a, b, p_1, p_2,\) and \(y\). There are several ways to check which is correct, conditional on these values. One way in this example is to substitute the actual values of \(a, b, y,\) and \(p_1\) into the expression for \(x_1\) in Equation A.25 and check whether it is positive. If not, \(x_1 = 0.\)

### Conditions for Existence and Uniqueness

Although the Kuhn-Tucker method gives us a general means of formulating problems of finding constrained extrema, there is no guarantee that a solution to the Kuhn-Tucker formulation exists. Even if a solution does exist, there is no guarantee that it is unique.

In Section A.4, we summarized the sufficient conditions that guarantee the existence and uniqueness of solutions to unconstrained extrema problems. Now we would like some simple conditions guaranteeing both the existence and the uniqueness of a solution to the Kuhn-Tucker formulation of a constrained extremum problem.

We want to specify these conditions for a general Kuhn-Tucker problem with \(n\) choice variables \(x_1, x_2, \ldots, x_n\), where we want to maximize an objective function \(f : \mathbb{R}^n \to \mathbb{R}\) subject to \(m\) constraints, \(g_j(x_1, x_2, \ldots, x_n) \geq 0\) for \(j = 1, 2, \ldots, m\):

\[
\max_{x_1, x_2, \ldots, x_n} f(x_1, x_2, \ldots, x_n) \\
\text{s.t. } g_j(x_1, x_2, \ldots, x_n) \geq 0, \quad \text{for} \quad j = 1, 2, \ldots, m.
\]  

The Slater condition guarantees the existence of a solution to problem A.28. The Slater condition requires that the solution to the maximization problem is not determined entirely by the constraints for any of the choice variables: There exists a point \((x_1, x_2, \ldots, x_n)\) such that \(g_j(x_1, x_2, \ldots, x_n) > 0\) for all \(j = 1, 2, \ldots, m\). Because this condition holds with a strict inequality, the constraint set has a non-empty interior.

A local maximum exists if the objective function and constraints are continuously differentiable and if the Slater condition is satisfied. If \((x_1^*, \ldots, x_n^*)\) is a local maximum of the problem A.28, it is also global maximum if \(f\) is weakly concave and if \(g_j\) is weakly convex for all \(j = 1, \ldots, m\). However, there could be more than one global maximum.

Sufficient conditions for a local maximum \((x_1^*, \ldots, x_n^*)\) to the problem A.28 to be a unique global maximum are that \(f\) is weakly concave; \(g_j\) is weakly convex for all \(j = 1, 2, \ldots, m\); and one of two alternative conditions holds:

1. The objective function \(f\) is strictly concave; or
2. At least one of the constraints \(g_j(x_1^*, \ldots, x_n^*) = 0\) and is strictly convex at \(g_j(x_1^*, \ldots, x_n^*).\)
The Envelope Theorem

We can state and prove a version of the Envelope Theorem that holds for constrained extremum problems. To facilitate this discussion, we use our previous formulation of the Kuhn-Tucker problem, but we explicitly add an exogenous parameter \( z \) so that \( z \) can have a direct effect on the objective function as well as a direct effect on any of the constraints \( g_i \).

\[
V(z) = \max_{x_1, x_2, \ldots, x_n} f(x_1, x_2, \ldots, x_n, z)
\]

s.t.

\[
g_j(x_1, x_2, \ldots, x_n, z) \geq 0, \quad \text{for} \quad j = 1, 2, \ldots, m,
\]

where \( V(z) \) is the maximized value of the objective function. The equivalent Lagrangian problem is

\[
V(z) = \max_{x_1, x_2, \ldots, x_n} f(x_1, x_2, \ldots, x_n, z) + \sum_{j=1}^{m} \lambda^j g_j(x_1, x_2, \ldots, x_n, z),
\]

where \( \lambda_1, \lambda_2, \ldots, \lambda_m \) are the Kuhn-Tucker multipliers.

The first-order conditions are

\[
\frac{\partial f}{\partial x_i} + \sum_{j=1}^{m} \lambda^j \frac{\partial g_j}{\partial x_i} = 0, \quad \text{for} \quad i = 1, \ldots, n \quad \text{and} \quad j = 1, \ldots, m,
\]

and the complementary slackness conditions are

\[
\lambda^j g(x_1, \ldots, x_n, z) = 0, \quad \text{for} \quad j = 1, \ldots, m.
\]

The Envelope Theorem states that, if the constraints \( g_i(x_1, x_2, \ldots, x_n, z) \) satisfy the Slater condition and if \( x_j(z) \), \( i = 1, 2, \ldots, n \), solve the first-order conditions, Equation A.31, and complementary slackness conditions, Equation A.32, then

\[
\frac{\partial V(z)}{\partial z} = \frac{\partial f(x_1, \ldots, x_n, z)}{\partial z} + \sum_{j=1}^{m} \lambda^j \frac{\partial g_j(x_1, \ldots, x_n, z)}{\partial z}.
\]

Proof. The value function \( V(z) = f(x_1(z), \ldots, x_n(z), z) + \sum_{j=1}^{m} \lambda^j g_j(x_1, \ldots, x_n, z) \). Differentiating this expression with respect to \( z \) yields

\[
\frac{\partial V(z)}{\partial z} = \frac{\partial f(x_1, \ldots, x_n, z)}{\partial z} + \sum_{i=1}^{n} \left[ \frac{\partial f(x_1, \ldots, x_n, z)}{\partial x_i} \frac{\partial x_i(z)}{\partial z} \right] + \sum_{j=1}^{m} \left[ \lambda^j \frac{\partial g_j(x_1, \ldots, x_n, z)}{\partial x_i} \frac{\partial x_i(z)}{\partial z} \right] \]

\[
+ \sum_{j=1}^{m} \left[ \frac{\partial \lambda^j(z)}{\partial z} g_j(x_1, \ldots, x_n, z) + \lambda^j(z) \frac{\partial g_j(x_1, \ldots, x_n, z)}{\partial z} \right].
\]

Collecting terms, we can rewrite this equation as

\[
\frac{\partial V(z)}{\partial z} = \frac{\partial f(x_1, \ldots, x_n, z)}{\partial z} + \sum_{j=1}^{m} \left[ \frac{\partial \lambda^j(z)}{\partial z} g_j(x_1, \ldots, x_n, z) + \lambda^j(z) \frac{\partial g_j(x_1, \ldots, x_n, z)}{\partial z} \right] \]

\[
+ \sum_{i=1}^{n} \left[ \frac{\partial f(x_1, \ldots, x_n, z)}{\partial x_i} \frac{\partial x_i(z)}{\partial z} \right] + \sum_{j=1}^{m} \left[ \lambda^j \frac{\partial g_j(x_1, \ldots, x_n, z)}{\partial z} \right] \frac{\partial x_i(z)}{\partial z}.
\]

\(^1\text{We could have added any finite number of such exogenous parameters; however, one is enough for our purposes.}\)
Using Equation A.31, the last bracketed expression in Equation A.33 equals zero. If we can show that the $\Sigma(\partial \lambda / \partial z)g_j$ expression in the other bracketed term is zero, we have proved the theorem. We know by the complementary slackness conditions that $\lambda_j g_j(x_1, x_2, \ldots, x_n, z) = 0$. If $g_j(x_1, x_2, \ldots, x_n, z) = 0$, then $(\partial \lambda_j / \partial z)g_j(x_1, x_2, \ldots, x_n, z) = 0$. Alternatively, if $g_j(x_1, x_2, \ldots, x_n, z) > 0$, so that $\lambda_j = 0$, the Slater condition implies that $\partial \lambda_j / \partial z = 0$, thus proving the theorem.

**Comparative Statics**

The method of comparative statics can often be applied when one is solving a problem with inequality constraints, but the matter is complicated by the need to keep track of which inequality constraints are binding. Let’s return to our earlier problem A.19, where the Lagrangian function is

$$\mathcal{L} = a \ln(x_1 + 1) + b \ln(x_2) + \lambda(y - p_1x_1 - p_2x_2) + \mu x_1$$

and has first-order conditions

$$\frac{a}{x_1 + 1} - \lambda p_1 + \mu = 0,$$

$$\frac{b}{x_2} - \lambda p_2 = 0,$$

and associated complementary slackness conditions

$$\lambda(y - p_1x_1 - p_2x_2) = 0, \quad (A.34)$$

$$\mu x_1 = 0. \quad (A.35)$$

These complementary slackness conditions complicate the comparative statics analysis. If a constraint is clearly binding, we don’t have a problem, because we know how it affects the solution. Unfortunately, we do not always know if a constraint binds.

In this example, we may be confident that the first constraint binds, so we know that $\lambda > 0$. Consequently, we can divide both sides of Equation A.34 by $\lambda$ to eliminate it from the complementary slackness conditions. However, we do not know whether the constraint $x_1 \geq 0$ is binding without knowing the actual parameters.

In one approach, we initially assume that all the constraints are binding, and then use this assumption to substitute the constraints into the first-order conditions and solve them. Here we assume that the constraint, Equation A.35, holds, $x_1 = 0$. Consequently, using Equation A.27, $x_2 = y/p_2$. Substituting these solutions into the first-order conditions, we have

$$a - \lambda p_1 + \mu = 0,$$

$$\frac{b}{y} - \lambda = 0,$$

or solving for $\mu$ and $\lambda$,

$$\mu = \frac{b}{y}p_1 - a,$$

$$\lambda = b/y.$$
Consequently, we’ve potentially solved the entire system, with proposed solutions for \( x_1, x_2 \), and both the multipliers. However, our initial assumption that \( x_1 = 0 \) implies that \( \mu > 0 \) or that \( (b/y)p_1 > a \). This last inequality is exactly what we need to check. If it's satisfied, then we have the correct solution that we’re at a corner. If it’s not, then the maximum is in the interior and not at a corner, and the constraint \( x_1 \geq 0 \) does not bind. If it’s not binding, then \( \mu = 0 \). Now we can go back and plug this condition into the first-order conditions, and solve. Given either set of these solutions, we can examine the effect of a change in a parameter.

**Recipe for Finding the Constrained Extrema of a Function**

The following is a step-by-step set of practical instructions for solving a constrained extrema problem. The focus of this section is very much on the mechanics of *how* rather than on the issues of *why*.

1. **Make the problem a maximization problem.** If the problem is to minimize \( f(x_1, x_2, \ldots, x_n) \) subject to constraints, we can convert it into a maximization problem by maximizing minus the function subject to the same constraints.

2. **Rewrite any constraints so that they take the form of “greater than or equal to zero.”** Here’s a brief field guide to constraints and how to deal with them:
   a. **Greater than or equal to:** If the constraint is initially stated in the form of \( g(x_1, x_2) \geq f(x_1, x_2) \), subtract the term \( f(x_1, x_2) \) from both sides to obtain \( g(x_1, x_2) - f(x_1, x_2) \geq 0 \).
   b. **Less than or equal to:** Given an initial constraint of \( g(x_1, x_2) \leq f(x_1, x_2) \), multiply both sides by \(-1\), making it a greater-than-or-equal-to problem, and use the method in (a): \( f(x_1, x_2) - g(x_1, x_2) \geq 0 \).
   c. **Strictly greater than or strictly less than:** If your constraint is \( g(x_1, x_2) > f(x_1, x_2) \) or \( g(x_1, x_2) < f(x_1, x_2) \), you’re in trouble! If this constraint has any effect on the problem, it will be to make it so that no solution exists. (Consider the problem of minimizing \( x \) such that \( x > 0 \) to see why this is a problem.) You have to reformulate your problem.
   d. **Equal to:** If it seems that the constraint truly has to hold with equality, put yourself in the shoes of the firm’s manager, who is doing the maximizing. If the firm could somehow challenge the natural order of things and violate the constraint, would the firm prefer a “less-than-or-equal-to” constraint or a “greater-than-or-equal-to” constraint? For example, a firm facing a constraint that required output \( q \) to be equal to a function \( f(x) \) of inputs \( x \) would, if the firm could violate the laws of nature, prefer that output was greater than production, or that \( q \geq f(x) \). Let’s give the firm the opposite of what it would want, imposing the constraint \( q \leq f(x) \). Multiply both sides by \(-1\) and then move the terms on the right-hand side to the left to get a “greater-than-or-equal-to-zero” constraint: \( f(x) - q \geq 0 \). Think again about whether the constraint really has to be an equality constraint—in this case, could the firm throw some output away? If the answer is yes, then you’re done. Otherwise, add another “greater-than-or-equal-to” constraint but with the opposite sign. So, in the example of our firm, we would have both constraints:
   \[
   f(x) - q \geq 0, \tag{A.36} \\
   q - f(x) \geq 0. \tag{A.37}
   \]
You can verify that these two inequality constraints imply a single equality constraint. Now you’ve got all your constraints formulated in the “greater-than-or-equal-to” form. Take a moment to be sure you haven’t neglected any. Are there some choice variables that can’t be negative? If so, add a non-negativity constraint requiring them to be greater than or equal to zero.

3. **Construct the Lagrangian function.** Assign a multiplier to each of your constraints (it’s traditional to use Greek letters for these multipliers), which you multiply times the left-hand side of the corresponding constraint, and add the products to the objective function you formulated in the first step.

4. **Partially differentiate the Lagrangian function.** Beginning with the first of your choice variables, partially differentiate the Lagrangian function with respect to this variable. Repeat for each of the remaining choice variables. Set each of these expressions equal to zero, yielding a collection of first-order conditions.

5. **List the complementary slackness conditions.** Take each of the products of the “greater-than-or-equal-to” constraints with their corresponding multipliers and set them equal to zero, yielding the complementary slackness conditions.

6. **Solve the system of equations.** Simultaneously solve the collection of first-order conditions and the complementary slackness conditions to find the critical values. The set of values of the choice variables that satisfy this system solve the constrained maximization problem.\(^2\)

\(^2\)We sometimes may be unable to find an explicit solution to these sets of equations, even if a solution exists. In such cases, we can use numerical techniques to find solutions (see Judd, 1998), or we can employ the method of comparative statics to try to understand the character of the solution.